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A SIMPLE PROOF OF NATAF'S THEOREM ON CONSISTENT AGGREGATION

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In this paper we present a simple and short proof of Nataf's theorem on consistent aggregation for the case that all functions involved have non-zero first derivatives.

1. The problem

The problem of consistent aggregation to be dealt with in this paper can most easily be illustrated by means of the scheme as indicated in fig. 1 [see also Pokropp (1972, p. 31)].

For ease of exposition we confine ourselves to production functions; this restriction, however, is not essential. Let x_{mj} be firm j's ($j = 1,\dots,J$) input of factor m ($= 1,\dots,M$). This firm's production function is f_j and its output is y_j. The outputs $y_1,\dots,y_J$ are aggregated into (an index of) aggregate production y by means of an aggregation function G. The question arises whether it is possible to get the same aggregate y if one takes the other way round: first aggregating all quantities of the production factors x_{mj} into aggregates x_m by means of aggregation functions g_m, for $m = 1,\dots,M$, respectively, and subsequently inserting these aggregates into an aggregate production function F.

In other words: the problem is whether there are functions f_j ($j = 1,\dots,J$), g_m ($m = 1,\dots,M$), F and G such that there exists a so-called

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$$
\begin{array}{cccccccc}
 & x_{11}, & \ldots, & x_{m1} & \ldots, & x_{M1} \xrightarrow{f_1} & & y_i \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & x_{1j}, & \ldots, & x_{mj}, & \ldots, & x_{Mj} \xrightarrow{f_j} & & y_j \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & x_{1J}, & \ldots, & x_{mJ}, & \ldots, & x_{MJ} \xrightarrow{f_J} & & y_J \\
g_1 \downarrow & & & g_m \downarrow & & g_M \downarrow & & G \downarrow \\
 & x_1, & \ldots, & x_m, & \ldots, & x_M \xrightarrow{F} & & y
\end{array}
$$

Fig. 1. Scheme of consistent aggregation.

atomistic macro function H of all x_{mj} such that for each matrix of quantities x_{mj} from a certain domain:

$$
\begin{aligned}
y &= H(x_{11},\ldots,x_{MJ}) \\
&= G\{f_1(x_{11},\ldots,x_{M1}),\ldots,f_J(x_{1J},\ldots,x_{MJ})\} \\
&= F\{g_1(x_{11},\ldots,x_{1J}),\ldots,g_M(x_{M1},\ldots,x_{MJ})\}.
\end{aligned}
\tag{1}
$$

The problem was posed by Klein (1946) and first solved by Nataf (1948) who proved the following:

Theorem. Let be given the functions of fig. 1 with non-zero first derivatives. For consistent aggregation in the sense of relation (1) it is necessary and sufficient that all the functions of the scheme are additively separable, as well as the atomistic macro function. This means that in case of consistent aggregation relation (1) turns into:

$$
y = \Phi\left\{ \sum_{m=1}^{M} \sum_{j=1}^{J} \phi_{mj}(x_{mj}) \right\},
\tag{2}
$$

where Φ *and the* ϕ_{mj} *are real-valued functions of a single variable.*

The above mentioned result is known as Nataf's theorem. The sufficiency of the condition of additive separability of all functions involved is obvious. Its necessity for consistent aggregation is less easier to prove. Nataf's proof, by means of first-order differential equations, may be called tedious. In (1964, pp. 35–39) Green gave a simpler, but rather lengthy proof of the necessity of additive separability of all functions of the scheme above assuming the existence of third-order derivatives. He did not prove this for the atomistic macro function, however. In Van Daal (1980, ch. 1) we presented a complete proof of Nataf's theorem, following Green's reasoning, under the weaker condition, however, of the existence of non-zero first derivatives. In this paper we present an alternative proof of Nataf's theorem under the latter condition which has the advantage of being much simpler and shorter. [1]

Our proof is based on two lemmas discussed in the next section. In section 3 the proof is given.

2. Two lemmas

Let f be a function of N variables that maps vectors of a certain domain $D \subset R^N$ into the real line R. Let $S=\{s_1,\ldots,s_K\}$ be a subset of $Z=\{1,\ldots,N\}$ and let $T=\{t_1,\ldots,t_L\}$ be its complement in Z; hence $K+L=N$. Now we define f to be *quasi-separable* with respect to S if there are functions ϕ and ψ of $L+1$ and K variables, respectively, such that

$$f(z_1,\ldots,z_N)=\phi\{\psi(z_{s_1},\ldots,z_{s_K}),x_{t_1},\ldots,z_{t_L}\}, \tag{3}$$

or, for short, in an obvious notation:

$$f(z^N)=\phi\{\psi(z^K),x^L\}.^{2} \tag{4}$$

In the remainder of this section f will always be a function as mentioned above.

It can be proved that the function f is quasi-separable with respect to

[1] Nataf's result, however, can also be proved under the condition of continuity only of the functions involved; see Gorman (1968). It can even be proved under the condition that the functions only meet some monotonicity conditions and may, therefore, have at most countably many points of discontinuity; see Pokropp (1972), and Somermeyer and Van Daal (1978). The proofs in Gorman (1968), Pokropp (1972) and Somermeyer and Van Daal (1978), as they are given under rather weak conditions, are fairly complicated.

[2] See Lemmas 1 and 2.

S if and only if for all s_k and $s_{k'} \in S$ the ratio $(\partial f/\partial z_{s_k})/(\partial f/\partial z_{s_{k'}})$ depends only on $z_{s_1},\ldots,z_{s_K}$; see Leontief (1947). See also Green (1964, pp. 12–15) for a proof where only the existence of non-zero first derivatives is used. From the just mentioned theorem we can easily derive [see Green (1964, proposition IVa)]:

Lemma 1. If $f(z_1,\ldots,z_N)$ is quasi-separable with respect to S as well as with respect to S', both subsets of $\{1,\ldots,N\}$ such that $S \cap S' \neq \phi$, then f is quasi-separable[3] *with respect to $S \cup S'$ as well as with respect to $S \cap S'$.*

The second lemma we need is on additive separability:

Lemma 2. The function f is additively separable, i.e., there are functions $g, h_1,\ldots,h_N$ of one variable such that for each z^N ($\in D \subset R^N$):

$$f(z_1,\ldots,z_N) = g\{h_1(z_1) + \ldots + h_N(z_N)\}, \tag{5}$$

if and only if for each z_n and $z_{n'}$ the ratio $(\partial f/\partial z_n)/(\partial f/\partial z_{n'})$ depends only on z_n and $z_{n'}$ for all $n, n' = 1,\ldots,N$.

For a proof of this lemma, using only the existence of non-zero first derivatives, we refer to Van Daal (1980).

3. Proof of Nataf's theorem

Since the sufficiency of the condition is obvious we only need to prove its necessity. This will be done in two stages.

(*a*) The additive separability of the atomistic macro function H.

To prove that relation (2) is the required specification for the atomistic macro function will consistent aggregation to be possible, we have to prove that for each pair x_{mj}, $x_{m'j'}$ the ratio $(\partial y/\partial x_{mj})/(\partial y/\partial x_{m'j'})$ depends only on x_{mj} and $x_{m'j'}$. This will be done in three steps (cf. fig. 2).

(i) With 'row j' and 'column m' will be denoted the sets $\{x_{1j},\ldots,x_{Mj}\}$ and $\{x_{m1},\ldots,x_{mJ}\}$, respectively. For consistent aggregation to be possible H must be quasi-separable with respect to row j as well as

[3] We use the term quasi-separable because in literature 'separability' mostly means that there are functions ϕ, ψ and x such that $f(z^N)=\phi\{\psi(z^K),\chi(z^L)\}$.

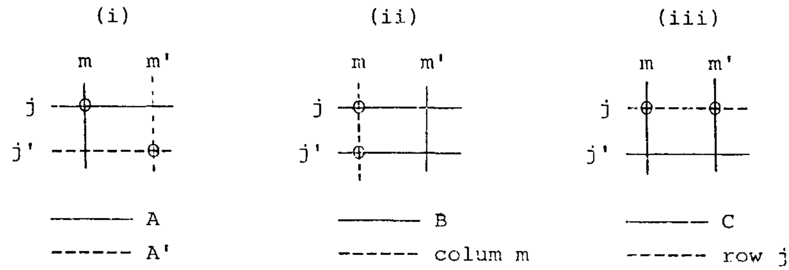

Fig. 2. Proof of the additive separability of H.

with respect to column m' for all $j = 1,\dots,J$ and $m' = 1,\dots,M$. From Lemma 1 it follows then that H is quasi-separable with respect to the union A of row j and column m' because their intersection is $\{x_{m'j}\}$ and is, therefore, non-void. In the same way we can argue that H is quasi-separable with respect to the union A' of row j' and column m. Application of Lemma 1 for the case that $m \neq m'$ and $j \neq j'$ yields that H is quasi-separable with respect to $A \cap A' = \{x_{mj}, x_{m'j'}\}$. This implies that $(\partial H/\partial x_{mj})/(\partial H/\partial x_{m'j'})$ depends only on x_{mj} and $x_{m'j'}$.

(ii) Let B be the union of A and row j'; their intersection is $\{x_{m'j'}\} \neq \phi$, hence H is quasi-separable with respect to B. Consequently, we conclude that H is quasi-separable with respect to the intersection of B and column m, being $\{x_{mj}, x_{mj'}\}$. Consequently, $(\partial H/\partial x_{mj})/(\partial H/\partial x_{mj'})$ depends only on x_{mj} and $x_{mj'}$.

(iii) Similarly it can be proved that for all m, m' and j $(\partial H/\partial x_{mj})/(\partial H/\partial x_{m'j})$ depends only on x_{mj} and $x_{m'j}$, viz. by considering the intersection of row j with the union C of A' and column m.

Because of Lemma 2, H is then additively separable, in other words: relation (2) holds.

(b) The additive separability of the other functions.

Because of the applicability of the scheme of fig. 1 and the additive separability of H we have for each $j = 1,\dots,J$ and $m = 1,\dots,M$:

$$\frac{\partial y}{\partial x_{mj}} = \frac{\partial G}{\partial y_j} \cdot \frac{\partial f_j}{\partial x_{mj}} = \Phi' \cdot \phi'_{mj}(x_{mj}), \tag{6}$$

where the primes attached to Φ and ϕ_{mj} denote differentiation. Dividing

(6) by the relation that we get if we replace m by m' in (6) yields that $(\partial f_j/\partial x_{mj})/(\partial f_j/\partial x_{m'j})$ equals $\phi'_{mj}(x_{mj})/\phi'_{m'j}(x_{m'j})$ and, therefore depends only on x_{mj} and $x_{m'j}$. Consequently, all micro functions f_j are additively separable.

Similarly we can prove that all aggregation functions g_m are additively separable.

The additive separability of F and G can be proved, without the use of Lemma 1, directly by means of Lemma 2. Using an argument that we owe to Green (1964, p. 37) [see also Leontief (1947)]:

$$\frac{\partial F/\partial x_m}{\partial F/\partial x_{m'}} = \frac{\partial f_j/\partial x_{mj}}{\partial g_m/\partial x_{mj}} \bigg/ \frac{\partial f_j/\partial x_{m'j}}{\partial g_{m'}/\partial x_{m'j}} \tag{7}$$

[differentiate (1) with respect to x_{mj} and $x_{m'j'}$ respectively, and divide the results]. This means that the ratios between $\partial F/\partial x_m$ and $\partial F/\partial x_{m'}$ depend at most on all micro variables $x_{m1},\dots,x_{mJ}$ and $x_{m'j},\dots,x_{m'J}$. Given consistent aggregation this means that the latter ratio depends only on the aggregates x_m and $x_{m'}$. Consequently, F is additively separable. In the same way this can be proved for G. This completes the proof of Nataf's theorem.

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